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*On the Calculation of Annuities, and on some Questions in the Theory of Chances.* By J. W. LUBBOCK, Esq., B.A.\*

[Extracted, by permission of the Author, from the *Transactions of the Cambridge Philosophical Society*.]

1. THE object of the following investigation is to show how the probabilities of an individual living any given number of years are to be deduced from any table of mortality. All writers (with the exception of Laplace) have considered the probability of an individual dying at any age to be the number of deaths at that age recorded in the table, divided by the sum of the deaths recorded at all ages. This would be the case if the observations on which the table is founded were infinite; but the supposition differs the more widely from the truth the less extended are the observations, and cannot, I think, be admitted where the recorded deaths do not altogether exceed a few thousand, as is the case in the tables used in England. The number of deaths on which the Northampton Tables are founded is 4,689 (Price, vol. i. p. 357). The tables of Halley are founded upon the deaths which took place at Breslau, in Silesia, during five years, and which amounted to 5,869.

If a bag contain an infinite number of balls of different colours in unknown proportions, a few trials or drawings will not indicate the proportion in which they exist in the bag, or the simple probability of drawing a ball of any given colour; and not only the probability of drawing a ball of any given colour, calculated from a few observations, will be little to be depended on, but it will also differ the more from the ratio of the number of times a ball of the given colour has been drawn, divided by the number of the preceding trials, the fewer the latter have been.

Laplace (*Théor. Anal. des Probabilités*, p. 426) has investigated the method of determining the value of annuities. He there says—"Si l'on nomme  $y_0$  le nombre des individus de l'âge A dans la table de mortalité dont on fait usage, et  $y_x$  le nombre des individus à l'âge  $A+x$ , la probabilité de payer la rente à la fin de l'année  $A+x$  sera  $\frac{y_x}{y_0}$ ." This hypothesis coincides with that I have before alluded to, as adopted by all other writers. Laplace, however, means this as an approximation, for he has investigated differently the probability of an individual of the age A living to the age  $A+a$  (p. 385 of the same work). He there considers two cases only possible; but as an individual may die at any instant

\* Now Sir John William Lubbock, Bart., F.R.S., &c.

during life, I think it may be doubted whether this hypothesis of possibility should be adopted.

Captain John Graunt was the first, if I am not mistaken, who directed attention to questions connected with the duration of life. He published a book in 1661, entitled *Observations on the Bills of Mortality*, which contains many interesting details, although it is written in the quaint style which prevailed in those times. In this book, amongst other tables there is one showing in 229,250 deaths how each arose; and another showing of 100 births "how many die within six years, how many the next decad, and so for every decad till 76"—which is in fact a table of mortality, and is probably the first ever published.

After Captain Graunt, Sir W. Petty published his *Essays on Political Arithmetick*. Halley, however, was the first who calculated tables of annuities: he took the probabilities on which they depend, from a table of mortality founded on the deaths during five years at Breslau. Since his time a great number of writers have treated of these subjects, of whom a notice may be seen in the *Encyclopædia Britannica*, or in the *Report from the Committee on the Laws respecting Friendly Societies*, 1827 (p. 94). It is to be regretted that those who have published tables of mortality should generally not only have altered the radix or number of deaths upon which the table is constructed, but also the number of deaths recorded at different ages, in order to render the decrements uniform; this is the case particularly with the Northampton Tables, as published by Dr. Price (*see Price on Reversionary Payments*, vol. i. p. 358). For if observations were continued to a sufficient extent, they would probably show that some ages are more exposed to disease than others—that is, they would indicate the existence of climacterics, of which alterations such as these destroy all trace.

I annex four tables,\* which I have calculated, with the assistance of Mr. Deacon, from the Tables of Mortality for Males and Females at Chester, given by Dr. Price (vol. ii. p. 392). The first two tables show the probability of an individual at any age living any given number of years, as well as the expectation of life at any age. The last two show the value of £1 to be received by an individual of any age after any number of years, and the value of an annuity. The difference between these values for a male and female is very great, and shows that tables which would be applicable for the one would not be for the other.

\* See page 207, and the note there.

I have also subjoined a table comparing the values of annuities calculated from observations at Chester (according to the hypothesis of probability I have assumed), with some which have been calculated from observations at other places. Until lately, the Government of this country granted annuities, the price of which depended on the price of stock, which renders their tables complicated. I have given their values of a deferred annuity for five years, compared with those I have calculated from the observations at Chester: it will be seen that the former are much too high.

2. Suppose a bag to contain a number of balls of  $p$  different colours, and that, having drawn  $m_1 + m_2 + m_3 \dots + m_p$  balls,  $m_1$  have been of the first colour,  $m_2$  of the second colour,  $m_3$  of the third colour,  $m_p$  of the  $p^{\text{th}}$  colour. If  $x_1, x_2, x_3 \dots x_p$  are the simple probabilities of drawing in one trial a ball of any given colour, the probability of the observed event is  $x_1^{m_1} \times x_2^{m_2} \dots \times x_p^{m_p}$ , multiplied by the coefficient of  $x_1^{m_1} x_2^{m_2} \dots x_p^{m_p}$  in the development of  $(x_1 + x_2 \dots + x_p)^{m_1 + m_2 \dots + m_p}$ . The event being observed, the probability of this system of probabilities is  $x_1^{m_1} \times x_2^{m_2} \dots \times x_p^{m_p}$ , divided by the sum of all possible values of this quantity.

The probability in  $n_1 + n_2 \dots + n_p$  subsequent trials of having  $n_1$  balls of the first colour,  $n_2$  of the second,  $n_p$  of the  $p^{\text{th}}$ , is a fraction of which the numerator is the sum of all the values of  $x_1^{m_1 + n_1} \times x_2^{m_2 + n_2} \dots \times x_p^{m_p + n_p}$ , and of which the denominator is the sum of all the values of  $x_1^{m_1} \times x_2^{m_2} \dots \times x_p^{m_p}$ , multiplied by the coefficient of  $x_1^{n_1} \times x_2^{n_2} \dots \times x_p^{n_p}$  in the development of  $(x_1 + x_2 + x_3 \dots + x_p)^{n_1 + n_2 \dots + n_p}$ .

Since  $x_1 + x_2 \dots + x_p = 1$ , if  $x_1, x_2$ , &c. be all supposed to vary from 0 to 1, and all these values to be equally possible *a priori*, the numerator will be found by integrating the expression

$$x_1^{m_1 + n_1} \times x_2^{m_2 + n_2} \dots (1 - x_1 - x_2 - x_3 \dots - x_{p-1})^{m_p + n_p} dx_1 \times dx_2 \dots \times dx_{p-1}$$

first from  $x_{p-1} = 0$  to  $x_{p-1} = 1 - x_1 - x_2 \dots - x_{p-2}$ , then from  $x_{p-2} = 0$  to  $x_{p-2} = 1 - x_1 \dots - x_{p-3}$ , and so on. The denominator will be found in the same way.

If the coefficient of  $x_1^{m_1} \times x_2^{m_2} \dots \times x_p^{m_p}$  in the development of  $(x_1 + x_2 \dots + x_p)^{n_1 + n_2 \dots + n_p}$  be called C, these integrations give for the probability required

$$C \times \frac{(m_1 + 1)(m_1 + 2)(m_1 + 3) \dots (m_1 + n_1)(m_2 + 1)(m_2 + 2) \dots (m_2 + n_2) \dots}{(m_1 + m_2 + m_3 \dots + m_p + p)(m_1 + m_2 + m_3 \dots + m_p + p + 1) \dots} \\ \frac{(m_p + 1)(m_p + 2) \dots (m_p + n_p)}{(m_1 + m_2 \dots + p + n_1 + n_2 + n_3 - 1)};$$

or if the product  $(m_p + 1)(m_p + 2) \dots (m_p + n_p)$  be denoted by  $[m_p + 1]^{n_p}$ , which is the notation used by Lacroix (*Traité du Calcul Différentiel*, vol. iii. p. 121), the probability required is

$$C \times \frac{[m_1 + 1]^{n_1} [m_2 + 1]^{n_2} \dots [m_p + 1]^{n_p}}{[m_1 + m_2 \dots m_p + p]^{n_1 + n_2 + n_3 \dots + n_p}}.$$

This probability is the same as if the simple probability of drawing a ball of the  $p^{\text{th}}$  colour were  $m_p + 1$ , with the difference of notation.

When  $n_2, n_3, n_{p-1}, \&c. = 0$ , and  $n_p = 1$ , this expression gives for the chance of drawing a ball of the  $p^{\text{th}}$  colour

$$\frac{m_p + 1}{m_1 + m_2 \dots + m_p + p},$$

and the probability that the index of the colour drawn is between  $n - 1$  and  $n + q + 1$  is

$$\frac{m_n + m_{n+1} \dots m_{n+q} + q}{m_1 + m_2 \dots + m_p + p}.$$

If we suppose the law of the possibility of life to be such that  $p$  cases or ages are possible—*à priori*,  $m_1, m_n, \&c.$  will be the number of recorded deaths in a table of mortality at those respective ages, and the chance of an individual living beyond the  $n^{\text{th}}$  age will be

$$\frac{m_n + m_{n+1} \dots m_p + p - n}{m_1 + m_2 \dots + m_p + p}.$$

$m_n + m_{n+1} + \&c. + m_p$  is the number given by the table as living at the  $n^{\text{th}}$  year; therefore, on the hypothesis of this law of possibility, the chance of an individual living beyond the  $n^{\text{th}}$  year is a fraction of which the numerator is the number living at that age,  $+p - n$ , and the denominator is the whole population on which the table is founded, or the radix  $+p$ . The Tables I. and II. have been calculated from this formula, from observations at Chester given by Dr. Price (vol. ii. p. 107):  $p$  was taken equal to 101 for a child at birth—that is, the chances of a child living beyond a hundred years, and of its dying in each intermediate year, were supposed to vary from 0 to 1, all these values being equally probable, *à priori*. The value of any sum to be received after any number of years is equal to the sum itself, multiplied by the chance of the individual being alive to receive it: therefore these tables give the value of unity to be received after any number of years. Considering duration of life to be valuable in proportion to its

length, the value of the expectation of life to any individual is the sum of the chances of his living any number of years multiplied by the intervening time; so that if  $P_n$  be the chance of an individual living *exactly*  $n$  years, the value of his expectation of life is  $\Sigma nP_n$ , which is evidently equal to  $\Sigma P'_n$ , if  $P'_n$  be the chance of an individual *surviving*  $n$  years: therefore the value of the expectation of life of any individual is the sum of the numbers on the same line in Tables I. and II. The unity of expectation is here the expectation of an individual who is certain to live exactly one year. The Tables I. and II. give the values of contingencies depending on a single life, without discount; the Tables III. and IV. are the same values, discounted at the rate of three per cent. compound interest. These tables give the values of annuities about six per cent. higher than those calculated from the Northampton, and given by Dr. Price, vol. ii. p. 54. The only tables of annuities on female lives that I have met with are calculated from observations in Sweden, and are given by Dr. Price, vol. ii. p. 422; but they are calculated at four and five per cent. interest. It is not to be expected, however, that tables calculated from observations made in one country will serve in another, or even in different parts of the same country.\*

The probability of having  $n_1$  balls of the first colour in  $n_1 + N$  trials, the colours of the other  $N$  balls being any whatever, is

$$\frac{\int x_1^{n_1+n_1}(1-x_1)^N x_2^{m_2} x_3^{m_3} \dots (1-x_1-x_2 \dots x_{p-1})^{m_p} dx_1 dx_2 \dots dx_{p-1}}{\int x_1^{m_1} x_2^{m_2} \dots (1-x_1-x_2 \dots x_{p-1})^{m_p} dx_1 dx_2 \dots dx_{p-1}},$$

multiplied by the coefficient of  $x^{n_1}$  in the development of  $(x_1 + y)^{n_1 + N}$ , the integrals being taken between the same limits as before.

These integrations give for the probability required

$$C \times \frac{(m_1+1)(m_1+2) \dots (m_1+n_1)(m_2+m_3+m_4 \dots + p-1)(m_2+m_3+m_4 \dots + p) \dots}{(m_1+m_2 \dots + m_p + p)(m_1+m_2 \dots + m_p + p+1) \dots} \cdot \frac{(m_2+m_3+m_4 \dots + p+N-2)}{(m_1+m_2+m_3 \dots + p+n_1+N-1)},$$

$C$  being equal to  $\frac{(n_1+1)}{1} \cdot \frac{(n_1+2)}{2} \dots \frac{(n_1+N)}{N+1}$ . Adopting the same notation as before, this probability is equal to

\* Since writing the above, I find that Mr. Finlaison has given the values of annuities, distinguishing the sexes, in the *Report of the Committee on Friendly Societies*, 1825, p. 140.

$$C \times \frac{[m_1+1]^{n_1} [m_2+m_3 \dots + m_p + p - 1]^N}{[m_1+m_2+m_3 \dots + m_p + p]^{n_1+N}}$$

$$= \frac{C[m_1+1]^{n_1} [m_2+m_3 \dots + m_p + p - 1]^{m_1+1}}{[m_2+m_3 \dots + N + p - 1]^{n_1+m_1+1}},$$

which probability, as before, is the same as if the simple probability of drawing a ball of the  $p^{\text{th}}$  colour were  $m_p + 1$ .

If  $m_2 + m_3 \dots + m_p + p - 2 = M$ , and if  $n_1$  and  $N$  are in the same ratio as  $m_1$  and  $M$ , the chance that the number of balls of the first colour in  $n_1 + N$  trials is between the limits  $n_1$  and  $n_1 \pm z$ , by the reductions given in the *Théorie Anal. des Probabilités*, p. 386, is

$$1 - 2 \sqrt{\frac{(M+m_1)^3}{m_1 M (N+n_1) (M+N+m_1+n_1) 2\pi}} \int dz e^{-\frac{\Gamma(M+m_1)^3 z^2}{2m_1 M (N+n_1) (M+N+m_1+n_1)}},$$

$e$  being the number of which the hyperbolic logarithm is unity, and the integral being taken from  $z = z$ , to  $z = \text{infinity}$ .

The question of determining the probability that the losses and gains of an Insurance Company on any class of life are contained within certain limits, is precisely similar to this.

It will be seen from the formula  $\frac{m_n + m_{n+1} \dots m_{n+q} + q}{m_1 + m_2 \dots + m_p + p}$  (p. 200, line 12), that if life were divided into an infinite number of ages or intervals (in which case  $p$  is infinite), the hypothesis of possibility remaining the same, the probability of an individual dying in any given interval would be the given interval divided by the whole duration of life, which coincides with that which is given by De Moivre's hypothesis. Thus if life were supposed to extend to a hundred years, the probability of an individual dying in any given year would be  $\frac{1}{100}$ , and any finite number of observations or recorded deaths would not influence the value of this probability. As diseases and other causes producing death are not equally distributed throughout life, the last hypothesis cannot be adopted.

In order to investigate accurately the probability of death at any age, it would be necessary to know the law of possibility. Let  $\phi_p x_p$  be the probability of the possibility of  $x_p$ : then the probability in the former question of having  $n_1$  balls of the first colour,  $n_2$  of the second, &c., in  $n_1 + n_2 \dots + n_p$  trials, is

$$C \times \frac{\int x_1^{m_1+n_1} (\phi_1 x_1) x_2^{n_2+n_2} (\phi_2 x_2) \dots (1-x_1-x_2 \dots x_{p-1})^{m_p+n_p} dx_1 dx_2 \dots dx_{p-1}}{\int x_1^{m_1} (\phi_1 x_1), x_2^{m_2} (\phi_2 x_2) \dots (1-x_1-x_2 \dots x_{p-1})^{m_p} dx_1 dx_2 \dots dx_{p-1}};$$

$\phi$  is a sign of function, and this function may be either continuous or discontinuous.

This expression must be integrated between the same limits as before.

The coefficients of the different powers of  $x_p$  in  $\phi_p x_p$ , or the constants in  $\phi_p x_p$ , will generally be functions of the index  $p$ . If the probability of life were known at a great many places, and if  $x_{p_1}$  were the value of  $x_p$  at  $q_1$  places,  $x_{p_2}$  at  $q_2$  places, &c., the law of possibility might be determined approximately by considering  $\phi_p x_p$  as a parabolic curve, of which  $x_p$  is the abscissa, passing through the points, of which the ordinates are

$$\frac{q_1}{q_1 + q_2 + \&c.}, \quad \frac{q_2}{q_1 + q_2 + \&c.}.$$

3. In the preceding investigations, the results of the preceding trials are supposed to be known; it may be worth while to examine what the probability of any future event is when the results of the preceding trials are uncertain.

Let a bag contain any number of balls of two colours, white and black: suppose  $m$  trials have taken place; and let  $e_n$  be the probability that a white ball was drawn the  $n^{\text{th}}$  trial,  $f_n$  the probability that a black ball was drawn.

$$e_n + f_n = 1.$$

First let  $e_1, e_2, \dots, e_n$  be all equal, and let  $x$  be the probability of drawing a white ball. If a white ball was drawn every time in the  $m$  trials which have taken place, the probability in  $n_1 + n_2$  future trials of having  $n_1$  white balls and  $n_2$  black balls is

$$\frac{(n_1 + n_2)(n_1 + n_2 - 1) \dots (n_1 + 1)}{1.2 \dots n_n} \frac{\int x^{m+n_1} (1-x)^{n_2} dx}{\int x^m dx}.$$

But the probability that a white ball was drawn every time is  $e^m$ ; therefore the probability of drawing a white ball  $n_1$  times and a black ball  $n_2$  times, on this hypothesis, multiplied by the probability of the hypothesis, is

$$\frac{(n_1 + n_2)(n_1 + n_2 - 1) \dots (n_1 + 1)}{1.2 \dots n_2} e^m \frac{\int x^{m+n_1} (1-x)^{n_2} dx}{\int x^m dx};$$

and the probability of drawing  $n_1$  white balls and  $n_2$  black balls will be the sum of the probabilities on every hypothesis, multiplied respectively by the probability of the hypothesis, which is

$$\frac{(n_1+n_2)(n_1+n_2-1)\dots(n_1+1)}{1.2\dots\dots n_2} \left\{ e^m \frac{\int x^{m+n_1}(1-x)^{n_2} dx}{\int x^m dx} \right. \\ \left. + m e^{m-1} \int \frac{\int x^{m+n_1-1}(1-x)^{n_2+1} dx}{\int x^{m-1}(1-x) dx} + \frac{m.m-1}{1.2} e^{m-2} \int \frac{\int x^{m+n_1-2}(1-x)^{n_2+2} dx}{\int x^{m-2}(1-x^2) dx} +, \&c. \right.$$

This integral being taken from  $x=0$  to  $x=1$ , is

$$\frac{(n_1+n_2)(n_1+n_2-1)\dots(n_1+1)}{1.2\dots\dots n_2} \left\{ \frac{n_2, n_2-1, n_2-2\dots\dots 1.m+1}{m+n_1+1.m+n_1+2\dots m+n_1+n_2+1} e^m \right. \\ \left. + m e^{m-1} f. \frac{n_2+1.n_2.n_2-1\dots 2}{m+n_1.m+n_1+1\dots m_1+n_1+n_2} m+1.m+, \&c. \right. \\ = \frac{(n_1+n_2)(n_1+n_2-1)\dots(n_1+1)}{1.2\dots\dots n_2} \times \frac{1}{m+2.m+3\dots m+n_1+n_2+1} \\ \left\{ n_2.n_2-1.n_2-2\dots m+n_1.m+n_1-1\dots m+1.e^m+n_2+1\dots \right. \\ \left. 2.m+n_1-1\dots m+1.m.m e^{m-1} f+, \&c. \right.$$

This series is equal to  $\frac{d^{n_1+n_2}.y^{n_2}x^{n_1}(ex+fy)^m}{dx^{n_1}dy^{n_2}}$ , when  $x$  and  $y$  are made equal to 1, and this is equal to  $1.2.3\dots n_1.1.2.3\dots n_2 \times$  coefficient of  $h^{n_1}k^{n_2}$ , in the development of

$$(1+h)^{n_1}(1+k)^{n_2}(1+eh+fk)^m \\ (1+eh+fk)^m = 1+m(eh+fk) + \frac{m.m-1}{1.2}(eh+fk)^2 \\ + \frac{m.m-1.m-2}{1.2.3}(eh+fk)^3+, \&c. \\ (1+h)^{n_1}(1+k)^{n_2} = h^{n_1}k^{n_2} + n_1h^{n_1-1}k^{n_2} + \frac{n_1.n_1-1}{1.2}h^{n_1-2}k^{n_2} \\ + \frac{n.n-1.n-2}{1.2.3}h^{n_1-3}k^{n_2}+, \&c. \\ + n_2h^{n_1}k^{n_2-1} + n.n_2h^{n_1-1}k^{n_2-1} + \frac{n_2.n_1.n_1-1}{1.1.2}h^{n_1-2}k^{n_2-1}+, \&c. \\ + \frac{n_2.n_2-1}{1.2}h^{n_1}k^{n_2-2} + \frac{n_1.n_2.n_2-1}{1.2}h^{n_1-1}k^{n_2-2}+, \&c. \\ + \frac{n_2.n_2-1.n_2-2}{1.2.3}h^{n_1}k^{n_2-3}+, \&c.$$

Coefficient of  $h^{n_1}k^{n_2} = 1+m(n_1e+n_2f)$

$$+ \frac{m.m-1}{1.2} \left( \frac{n_1.n_1-1}{1.2} e^2 + 2n_1n_2ef + \frac{n_2.n_2-1}{1.2} f^2 \right) +, \&c.$$

The probability required is

$$\frac{1.2.3.\dots.n_1+n_2}{m+2.m+3.\dots.m+n_1+n_2+1} \left\{ 1+m(n_1e+n_2f)+\frac{m.m-1}{1.2} \left\{ \frac{n_1.n_1-1}{1.2}e^2+2n_1n_2ef+\frac{n_2.n_2-1}{1.2}f^2 \right\} \right\} +, \&c.$$

If there are  $p$  different colours, and if  $m$  trials have taken place, and  $e_{q,p}$  is the chance that a ball of the  $p^{\text{th}}$  colour was drawn the  $q^{\text{th}}$  trial, the probability of drawing  $n_1$  balls of the first colour,  $n_2$  of the second,  $n_p$  of the  $p^{\text{th}}$ , in  $n_1+n_2+\dots+n_p$  future trials, may be found in the same way. Let

$$e_{1,1}+e_{1,2}+e_{1,3}+\&c.\dots.e_{1,n}=S_1, e_1,$$

$$e_{1,1}, e_{1,2}+e_{1,3}, e_{1,4}+\dots+\&c.=S_2, e_1,$$

(the sum of the products of  $e_1$  two and two together,)

$$e_{1,1}, e_{2,2}+e_{1,2}, e_{2,3}+\dots = S_1e_1, S_1e_2,*$$

and so on; then it may be shown that this probability is equal to

$$\frac{1.2.3.\dots.n_1+n_2+n_3+\dots+n_p}{m+p.\dots.m+n_1+n_2+\dots+n_p+p-1} (1+Se_1)^{n_1}(1+Se_2)^{n_2}\dots(1+Se_p)^{n_p},$$

$1+(Se_1)$ ,  $1+(Se_2)$ , &c. being expanded by the binomial theorem, and the indices of  $S$  written at the foot.

The method which was used for summing the series in the last page is of very general application, and depends, in fact, on this principle, that the generating function of the sum of any series is the sum of the generating functions of each of the terms of the series.

If in the last formula  $n_2, n_3, \&c.=0$ , and if there be only two events possible, and  $n_1=1$ , the probability required is  $\frac{1+Se_1}{m+2}$ . In order to apply this, suppose an individual to have asserted  $m$  events to have taken place, of which the simple probabilities are equal, and equal to  $p$ ; and suppose it required to find the probability of his telling the truth in another case, where the simple probability of the event he asserts to have taken place is not known. Let  $x$  be the veracity of the individual, the probability of his telling the truth on this hypothesis is  $\frac{px}{px+(1-x)(1-p)}$ ; and the probability of his telling the truth is the sum of the probabilities of his telling the truth on each hypothesis, divided by the number of the hypotheses.

\* This is a method of notation which obtains, but it is not meant to imply that  $S_1e_1S_1e_2=S_1,e_1 \times S_1,e_2$ .

Suppose  $x$  to vary from 0 to 1, and all these values of  $x$  to be equally probable *à priori*, then the probability of his having told the truth and the event having taken place is  $\int \frac{p x d x}{p x+(1-p)(1-x)}$ , taken from  $x=0$  to  $x=1$ , which integral is

$$\frac{p}{2 p-1}\left\{1-\frac{(1-p)}{2 p-1} \text { hyp. log. } \frac{p}{1-p}\right\} .$$

If  $p=\frac{9}{10}$ , this probability is .81601. Generally, if  $p>\frac{1}{2}$ , the assertion that the event has taken place (on this hypothesis of veracity) rather diminishes the probability that the event has taken place; if  $p=\frac{1}{2}$ , the assertion does not alter the probability; if  $p<\frac{1}{2}$ , the assertion rather increases it.

If  $p=\frac{9}{10}$ ,  $e=.81601$ , let  $m=10$ ; then  $\frac{1+Se}{m+2}=\frac{9.1601}{12}$ , which is the probability that the individual will tell the truth in another case. If the individual had told ten truths, the chance of his telling the truth in another case would have been  $\frac{11}{12}$ .

All values of  $x$  between 0 and 1 were supposed equally possible: if they are not, let  $\phi x$  be the probability of the possibility of any value of  $x$ ; then the probability of an individual telling the truth will be  $\frac{\int \phi x d x}{\int \phi x d x+(1-p) \int \phi x d x}$  divided by  $\int \phi x d x$ , these integrals being taken from  $x=0$  to  $x=1$ .

Table formed from the Burials in All Saints' Parish, Northampton, from 1735 to 1780. (See page 198, line 26.)

AGE.	Actual Number of Burials.	Reduced to radix 11650.	
		By the Observations.	As altered by Dr. Price.
Under . . 2	1,529	3798 $\frac{1}{2}$	4367
Between 2 and 5	362	899 $\frac{1}{2}$	1034
"      5 " 10	201	499 $\frac{3}{4}$	574
"     10 " 20	189	469 $\frac{1}{2}$	543
"     20 " 30	373	926 $\frac{1}{2}$	747
"     30 " 40	329	817 $\frac{1}{2}$	750
"     40 " 50	365	906 $\frac{1}{2}$	778
"     50 " 60	384	954	819
"     60 " 70	378	939 $\frac{1}{2}$	806
"     70 " 80	358	889	763
"     80 " 90	199	494	423
"     90 " 100	22	54 $\frac{1}{2}$	46
TOTAL . . .	4,689	11650	11650

TABLE I.—*Males.*

Age.	Expectation of Life, by the Author's method.	Expectation of Life, by usual method.
0	29·75345	28·13
1	36·69540	35·76
2	40·21306	39·42
3	42·54615	41·97
4	43·83928	43·33
5	44·09357	43·20
10	42·75204	41·92
15	38·95786	38·05
20	35·82561	34·86
25	32·99367	32·00
30	30·27118	29·25
35	27·04332	25·97
40	24·04172	22·92
45	21·34876	20·20
50	18·81444	17·64
55	16·34857	15·14
60	13·63392	12·36
65	12·05917	10·79
70	9·41263	8·05
75	8·43636	7·00
80	6·99009	5·43
85	5·90384	4·25
90	4·32000	2·50
95	2·14285	1·00

TABLE II.—*Females.*

Age.	Expectation of Life, by the Author's method.	Expectation of Life, by usual method.
0	34·55535	33·27
1	40·04475	39·54
2	43·65276	43·25
3	45·87700	45·68
4	47·23533	47·11
5	47·99860	47·44
10	45·69310	45·17
15	41·96030	41·36
20	38·76308	38·10
25	35·49329	34·78
30	32·79565	32·27
35	30·00384	29·26
40	27·13292	26·37
45	24·29072	23·50
50	21·43212	20·62
55	18·35900	17·52
60	15·09954	14·20
65	12·83834	11·94
70	9·78378	8·81
75	8·12794	7·14
80	6·30434	5·20
85	5·97402	4·81
90	4·55263	3·46
95	2·07692	1·71

TABLE III.—*Males.*

Age.	Value of Annuity by the Author's method.	Value of Annuity by usual method.
0	13·96256	
1	17·35468	
2	19·17322	
3	20·38907	
4	21·15972	
5	21·42118	21·283
10	21·55443	21·512
15	20·38198	20·283
20	19·42818	19·285
25	18·55566	18·399
30	17·67138	17·492
35	16·36473	16·134
40	15·05567	14·667
45	13·81590	13·493
50	12·59164	12·316
55	11·28375	10·866
60	9·63491	9·140
65	8·77709	8·220
70	6·94786	6·260
75	6·17140	5·291
80	5·11141	
85	4·30505	

TABLE IV.—*Females.*

Age.	Value of Annuity by the Author's method.	Value of Annuity by usual method.
0	15·75290	
1	18·42550	
2	20·14850	
3	21·32367	
4	22·11858	
5	22·64306	22·624
10	22·41169	22·439
15	21·25267	21·235
20	20·36435	20·323
25	19·33571	19·265
30	18·65687	18·583
35	17·62920	17·534
40	16·56779	16·366
45	15·42043	15·282
50	14·14872	13·983
55	12·58232	12·376
60	10·67266	10·410
65	9·37655	9·080
70	7·26287	6·889
75	6·02689	5·338
80	4·65536	
85	4·34792	

NOTE.—We do not reprint all the tables originally given in this paper. They are of value now only as exhibiting the results of the methods laid down by the Author, and for that purpose those here quoted will suffice.—ED. A. M.